

A Universal Action Formula

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Abstract. ■

A universal formula for an action associated with a noncommutative geometry, defined by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, is proposed. It is based on the spectrum of the Dirac operator and is a geometric invariant. The new symmetry principle is the automorphism of the algebra \mathcal{A} which combines both diffeomorphisms and internal symmetries. Applying this to the geometry defined by the spectrum of the standard model gives an action that unifies gravity with the standard model at a very high energy scale.

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Riemannian geometry has played an important role in our understanding of space-time especially in the development of the general theory of relativity. The basic data of Riemannian geometry consists in a *manifold* M whose points $x \in M$ are locally labelled by finitely many coordinates $x^\mu \in \mathbb{R}$, and in the infinitesimal *line element*, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. The dynamics of the metric is governed by the Einstein action and the symmetry principle is diffeomorphism invariance. The other basic interactions consisting of strong, weak and electromagnetic forces are well described by the standard model action and the symmetry principle is the gauged internal symmetry. Therefore the natural group of invariance that governs the sum of the Einstein and standard model actions is the semi-direct product of the group of local gauge transformations, $\mathcal{U} = C^\infty(M, U(1) \times SU(2) \times SU(3))$ by the natural action of $\text{Diff}(M)$. Another concept which is vital to the construction of the standard model is spontaneous symmetry breaking and Higgs fields but this has no geometrical significance.

In a new development in mathematics, it has been shown that Riemannian geometry could be replaced with a more general formulation, noncommutative geometry. The basic data of noncommutative geometry consists of an involutive *algebra* \mathcal{A} of operators in Hilbert space \mathcal{H} and of a selfadjoint unbounded operator D in \mathcal{H} [1-6]. The inverse D^{-1} of D plays the role of the infinitesimal unit of length ds of ordinary geometry.

There is no information lost in trading the original Riemannian manifold M for the corresponding spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M , $\mathcal{H} = L^2(M, S)$ the Hilbert space of L^2 -spinors and D the Dirac operator of the Levi-Civita Spin connection. More importantly one can characterize the spectral triples $(\mathcal{A}, \mathcal{H}, D)$ which come from the above spinorial construction by very simple axioms ([4]) which involve the dimension n of M . The parity of n implies a $\mathbb{Z}/2$ grading γ of the Hilbert space \mathcal{H} such that,

$$\gamma = \gamma^* , \quad \gamma^2 = 1 , \quad \gamma a = a \gamma \quad \forall a \in \mathcal{A} , \quad \gamma D = -D \gamma .$$

Moreover one keeps track of the *real structure* on \mathcal{H} as an antilinear isometry J in \mathcal{H} satisfying simple relations

$$J^2 = \varepsilon , \quad J D = \varepsilon' D J , \quad J \gamma = \varepsilon'' \gamma J ; \quad \varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$$

where the value of $\varepsilon, \varepsilon', \varepsilon''$ is determined by n modulo 8. The usual emphasis on the points $x \in M$ of a geometric space is now replaced by the spectrum $\Sigma \subset \mathbb{R}$ of

the operator D . Indeed, if one forgets about the algebra \mathcal{A} in the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ but retains only the operators D , γ and J acting in \mathcal{H} one can characterize this data by the spectrum Σ of D which is a discrete subset with multiplicity of \mathbb{R} . In the even case $\Sigma = -\Sigma$. The existence of Riemannian manifolds which are isospectral (i.e. have the same Σ) but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity,

“The physical action depends only on Σ .”

The most natural candidate for an action that depends only on the spectrum is

$$\text{Trace } \chi \left(\frac{D}{\Lambda} \right) + \langle \psi, D\psi \rangle. \quad (1)$$

where χ is a positive function. This form of the action is dictated by the condition that it is additive for disjoint union. In the usual Riemannian case the group $\text{Diff}(M)$ of diffeomorphisms of M is canonically isomorphic to the group $\text{Aut}(\mathcal{A})$ of automorphisms of the algebra $\mathcal{A} = C^\infty(M)$. To each $\varphi \in \text{Diff}(M)$ one associates the algebra preserving map $\alpha_\varphi : \mathcal{A} \rightarrow \mathcal{A}$ given by

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \quad \forall f \in C^\infty(M) = \mathcal{A}.$$

In general the group $\text{Aut}(\mathcal{A})$ of automorphisms of the involutive algebra \mathcal{A} plays the role of the diffeomorphisms of the noncommutative (or spectral for short) geometry $(\mathcal{A}, \mathcal{H}, D)$. The first interesting new feature of the general case is that the group $\text{Aut}(\mathcal{A})$ has a natural normal subgroup, $\text{Int}(\mathcal{A}) \subset \text{Aut}(\mathcal{A})$, where an automorphism α is *inner* iff there exists a unitary operator $u \in \mathcal{A}$, ($uu^* = u^*u = 1$) such that, $\alpha(a) = uau^* \quad \forall a \in \mathcal{A}$. This leads us to the postulate that: *The symmetry principle in noncommutative geometry is invariance under the group $\text{Aut}(\mathcal{A})$.*

We now apply these ideas to derive a noncommutative geometric action unifying gravity with the standard model. The algebra is taken to be $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra \mathcal{A}_F is *finite dimensional*, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and $\mathbb{H} \subset M_2(\mathbb{C})$ is the algebra of quaternions, $\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} ; \alpha, \beta \in \mathbb{C} \right\}$. \mathcal{A} is a tensor product which geometrically corresponds to a product space, an instance of spectral geometry for \mathcal{A} is given by the product rule,

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = \not{D}_M \otimes 1 + \gamma_5 \otimes D_F \quad (2)$$

where (\mathcal{H}_F, D_F) is a spectral geometry on \mathcal{A}_F , while both $L^2(M, S)$ and the Dirac operator \not{D}_M on M are as above.

The group $\text{Aut}(\mathcal{A})$ of diffeomorphisms falls in equivalence classes under the normal subgroup $\text{Int}(\mathcal{A})$ of inner automorphisms. In the same way the space of metrics has a natural foliation into equivalence classes. The *internal fluctuations* of a given metric are given by the formula,

$$D = D_0 + A + JAJ^{-1}, \quad A = \Sigma a_i [D_0, b_i], \quad a_i, b_i \in \mathcal{A} \text{ and } A = A^*. \quad (3)$$

Thus starting from $(\mathcal{A}, \mathcal{H}, D_0)$ with obvious notations, one leaves the representation of \mathcal{A} in \mathcal{H} untouched and just perturbs the operator D_0 by (3) where A is an arbitrary self-adjoint operator in \mathcal{H} of the form $A = \Sigma a_i [D_0, b_i]$; $a_i, b_i \in \mathcal{A}$. For Riemannian geometry these fluctuations are trivial.

The hypothesis which we shall test in this letter is that there exist an energy scale Λ in the range $10^{15} - 10^{19}$ GeV at which we have a geometric action given by (1).

$$\text{The quarks } Q \text{ and leptons } L \text{ are : } Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix}, \quad L = \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix}.$$

We now describe the internal geometry. The choice of the Dirac operator and the action of \mathcal{A}_F in \mathcal{H}_F comes from the restrictions that these must satisfy:

$$\begin{aligned} J^2 &= 1, \quad [J, D] = 0, \quad [a, Jb^* J^{-1}] = 0 \\ [[D, a], Jb^* J^{-1}] &= 0 \quad \forall a, b. \end{aligned} \quad (4)$$

We can now compute the inner fluctuations of the metric and thus operators of the form: $A = \Sigma a_i [D, b_i]$. This with the self-adjointness condition $A = A^*$ gives a $U(1)$, $SU(2)$ and $U(3)$ gauge fields as well as a Higgs field. The computation of $A + JAJ^{-1}$ removes a $U(1)$ part from the above gauge fields (such that the full matrix is traceless). The Dirac operator D_q that takes the inner fluctuations into account is given by the 36×36 matrix (acting on the 36 quarks) (tensored with Clifford algebras)

$$\begin{aligned} D_q &= \\ &\left[\begin{array}{ccc} \gamma^\mu \otimes (D_\mu \otimes 1_2 - \frac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha - \frac{i}{6} g_{01} B_\mu \otimes 1_2) \otimes 1_3, & \gamma_5 \otimes k_0^d \otimes H, & \gamma_5 \otimes k_0^u \otimes \tilde{H} \\ \gamma_5 \otimes k_0^{d*} \otimes H^*, & \gamma^\mu \otimes (D_\mu + \frac{i}{3} g_{01} B_\mu) \otimes 1_3, & 0 \\ \gamma_5 k_0^{u*} \tilde{H}^*, & 0, & \gamma^\mu \otimes (D_\mu - \frac{2i}{3} g_{01} B_\mu) \otimes 1_3 \end{array} \right] \otimes 1_3 \\ &+ \gamma^\mu \otimes 1_4 \otimes 1_3 \otimes \left(-\frac{i}{2} g_{03} V_\mu^i \lambda^i \right) \end{aligned} \quad (5)$$

where σ^α are Pauli matrices and λ^i are Gell-Mann matrices satisfying $\text{Tr}(\lambda^i \lambda^j) = 2\delta^{ij}$. The vector fields B_μ , A_μ^α and V_μ^i are the $U(1)$, $SU(2)_w$ and $SU(3)_c$ gauge fields with gauge couplings g_{01} , g_{02} and g_{03} . The differential operator D_μ is given by $D_\mu = \partial_\mu + \omega_\mu$ where $\omega_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ and $\gamma^\mu = e_a^\mu \gamma^a$. The scalar field H is the Higgs doublet, and $\tilde{H} = (i\sigma^2 H)$ is the $SU(2)$ conjugate of H .

The Dirac operator acting on the leptons, taking inner fluctuations into account is given by the 9×9 matrix (tensored with Clifford algebra matrices):

$$D_\ell = \begin{bmatrix} \gamma^\mu \otimes (D_\mu - \frac{i}{2}g_{02}A_\mu^\alpha \sigma^\alpha + \frac{i}{2}g_{01}B_\mu \otimes 1_2) \otimes 1_3 & \gamma_5 \otimes k_0^e \otimes H \\ \gamma_5 \otimes k_0^{*e} \otimes H^* & \gamma^\mu \otimes (D_\mu + ig_{01}B_\mu) \otimes 1_3 \end{bmatrix}. \quad (6)$$

The matrices k_0^d , k_0^u and k_0^e are 3×3 family mixing matrices. According to our universal formula the spectral action for the standard model is given by:

$$\text{Tr}[\chi(D^2/m_0^2)] + (\psi, D\psi) \quad (7)$$

where $(\psi, D\psi)$ will include both the quark and leptonic interactions. It is a simple exercise to compute the square of the Dirac operator given by (5) and (6). This can be cast into the elliptic operator form [7]:

$$P = D^2 = -(g^{\mu\nu} \partial_\mu \partial_\nu \cdot \mathbb{I} + \mathbb{A}^\mu \partial_\mu + \mathbb{B})$$

where \mathbb{I} , \mathbb{A}^μ and \mathbb{B} are matrices of the same dimensions as D . Using the heat kernel expansion for

$$\text{Tr} e^{-tP} \simeq \sum_{n \geq 0} t^{\frac{n-m}{d}} \int_M a_n(x, P) dv(x)$$

where m is the dimension of the manifold in $C^\infty(M)$, d is the order of P (in our case $m = 4$, $d = 2$) and $dv(x) = \sqrt{g} d^m x$ where $g^{\mu\nu}$ is the metric on M appearing in P , we can show that

$$\text{Tr} \chi(P) \simeq \sum_{n \geq 0} f_n a_n(P)$$

where the coefficients f_n are given by

$$\begin{aligned} f_0 &= \int_0^\infty \chi(u) u du, \quad f_2 = \int_0^\infty \chi(u) du, \\ f_{2(n+2)} &= (-1)^n \chi^{(n)}(0), \quad n \geq 0 \end{aligned}$$

and $a_n(P) = \int a_n(x, P) dv(x)$ are the Seeley-de Witt coefficients. $a_n(P)$ vanish for odd values of n and the first four a_n for even n are given in Gilkey[7]. A very lengthy but straightforward calculation, the details of which will be reported somewhere else [8] gives for the bosonic action

$$\begin{aligned}
I_b = \int d^4x \sqrt{g} \left[\frac{1}{2\kappa_0^2} R - \mu_0^2 (H^* H) + a_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right. \\
+ b_0 R^2 + c_0 {}^*R^*R + d_0 R_{;\mu}{}^\mu \\
+ e_0 + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \\
\left. + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_\mu H|^2 - \xi_0 R |H|^2 + \lambda_0 (H^* H)^2 \right] + O\left(\frac{1}{m_0^2}\right) \quad (8)
\end{aligned}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and ${}^*R^*R$ is the Euler characteristic, and

$$\begin{aligned}
\mu_0^2 &= \frac{4}{3\kappa_0^2} & a_0 &= -\frac{9}{8g_{03}^2} & b_0 &= 0 \\
c_0 &= -\frac{11}{18} a_0 & d_0 &= -\frac{2}{3} a_0 & e_0 &= \frac{45}{4\pi^2} f_0 m_0^4 \\
\lambda_0 &= \frac{4}{3} g_{03}^2 \frac{z^2}{y^4} & \xi_0 &= \frac{1}{6}.
\end{aligned} \quad (9)$$

We have also denoted

$$\begin{aligned}
y^2 &= \text{Tr} \left(|k_0^d|^2 + |k_0^u|^2 + \frac{1}{3} |k_0^e|^2 \right) \\
z^2 &= \text{Tr} \left((|k_0^d|^2 + |k_0^u|^2)^2 + \frac{1}{3} |k_0^e|^4 \right) \\
D_\mu H &= \partial_\mu H - \frac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha H - \frac{i}{2} g_{01} B_\mu H.
\end{aligned}$$

The Einstein, Yang-Mills and Higgs term are normalized by taking:

$$\frac{15m_0^2 f_2}{4\pi^2} = \frac{1}{\kappa_0^2} \quad \frac{g_{03}^2 f_4}{\pi^2} = 1 \quad g_{03}^2 = g_{02}^2 = \frac{5}{3} g_{01}^2. \quad (10)$$

Relations (10) among the gauge coupling constants coincide with those coming from $SU(5)$ unification.

We shall adopt Wilson's view point of the renormalization group approach to field theory [9] where the spectral action is taken to give the *bare* action with bare quantities $a_0, b_0, c_0 \dots$ and at a cutoff scale Λ which regularizes the action

the theory is assumed to take a geometrical form. The perturbative expansion is then reexpressed in terms of *renormalized* physical quantities. The fields also receive wave function renormalization.

The renormalized action receives counterterms of the same form as the bare action but with physical parameters. a, b, c, \dots . The renormalization group equations will yield relations between the bare quantities and the physical quantities with the addition of the cutoff scale Λ . Conditions on the bare quantities would translate into conditions on the physical quantities. The renormalization group equations of this system were studied by Fradkin and Tseytlin [10] and is known to be renormalizable, but non-unitary [11] due to the presence of spin-two ghost (tachyon) pole near the Planck mass. We shall not worry about non-unitarity (see, however, reference 12), because in our view at the Planck energy the manifold structure of space-time will break down and must be replaced with a genuinely noncommutative structure.

Relations between the bare gauge coupling constants as well as equations (3.19) have to be imposed as boundary conditions on the renormalization group equations [9]. The bare mass of the Higgs field is related to the bare value of Newton's constant, and both have quadratic divergences in the limit of infinite cutoff Λ . The relations between m_0^2, e_0 and the physical quantities are:

$$\begin{aligned} m_0^2 &= m^2 \left(1 + \frac{(\frac{\Lambda_2}{m^2} - 1)}{32\pi^2} \left(\frac{9}{4} g_2^2 + \frac{3}{4} g_1^2 + 6\lambda - 6k_t^2 \right) \right) + 0 \left(\ln \frac{\Lambda^2}{m^2} \right) + \dots \\ e_0 &= e + \frac{\Lambda^4}{32\pi^2} (62) + \dots \end{aligned} \quad (11)$$

For $m^2(\Lambda)$ to be small at low-energies m_0^2 should be tuned to be proportional to the cutoff scale according to equation (11). Similarly the bare cosmological constant is related to the physical one (which must be tuned to zero at low energies). There is also a relation between the bare scale κ_0^{-2} and the physical one κ^{-2} which is similar to equation (11) (but with all one-loop contributions coming with the same sign) which shows that $\kappa_0^{-1} \sim m_0$ and Λ are of the same order as the Planck mass.

The renormalization group equations for the gauge coupling constants g_1, g_2, g_3 are the same as those with $SU(5)$ boundary conditions, and can be easily solved using the present experimental values for $\alpha_{em}^{-1}(M_Z)$ and $\alpha_3(M_Z)$ to give [13]:

$$\Lambda \simeq 10^{15}(\text{Gev}) \quad \sin^2 \theta_w \simeq 0.210. \quad (12)$$

There is one further relation in our theory between the $\lambda_0(H^*H)^2$ coupling and the gauge couplings in equation (9) imposed at the scale Λ . This relation could be simplified if we assume that the top quark Yukawa coupling is much larger than all the other Yukawa couplings. In this case the equation simplifies to $\lambda(\Lambda) = \frac{16\pi}{3}\alpha_3(\Lambda)$. Therefore the value of λ at the unification scale is $\lambda_0 \simeq 0.402$ showing that one does not go outside the perturbation domain. In reality the RG equations for λ and k_t together with the boundary condition on λ could be used to determine the Higgs mass at the low-energy scale M_Z . The renormalization group equations are complicated and must be integrated numerically [14]. One can read the solution for the Higgs mass from the study of the triviality bound and this gives $m_H = 170 - 180$ Gev. One expects this prediction to be correct to the same order as that of $\sin^2 \theta_w$ which is off from the experimental value by ten percent. Therefore the bare action we obtained and associated with the spectrum of the standard model is consistent within ten percent provided the cutoff scale is taken to be $\Lambda \sim 10^{15}$ Gev at which the action becomes geometrical.

There is, however, a stronger disagreement where Newton's constant comes out to be too large. This is so because the gravity sector requires the cutoff scale to be of the same order as the Planck scale while the condition on gauge coupling constants give $\Lambda \sim 10^{15}$ Gev. One easy way to avoid this problem is to assume that the spectrum contains in addition a fermionic particle which only interacts gravitationally (such as a gravitino), but at present we shall not commit ourselves. Incidentally the problem that Newton's constant is coming out to be too large is also present in string theory where also has unification of gauge couplings and Newton's constant occurs [15]. These results must be taken as an indication that the spectrum of the standard model has to be altered as we climb up in energy. The change may happen at low energies (just as in supersymmetry which also pushes the cutoff scale to 10^{16} Gev) or at some intermediate scale. Ultimately one would hope that modification of the spectrum will increase the cutoff scale nearer to the Planck mass as dictated by gravity.

To summarize, we have shown that the basic symmetry for a noncommutative space $(\mathcal{A}, \mathcal{H}, D)$ is $\text{Aut}(\mathcal{A})$. This symmetry includes diffeomorphisms and internal symmetry transformations. The bosonic action is a spectral function of the Dirac operator while the fermionic action takes the simple linear form $(\psi, D\psi)$ where ψ are spinors defined on the Hilbert space. Applying this principle to the noncommutative geometry of the standard model gives the standard model action coupled to Einstein and Weyl gravity plus higher order non-renormalizable interac-

tions suppressed by powers of the inverse of the mass scale in the theory. There are some relations between the bare quantities. The renormalized action will have the same form as the bare action but with physical quantities replacing the bare ones. The relations among the bare quantities must be taken as boundary conditions on the renormalization group equations governing the scale dependence of the physical quantities. These boundary conditions imply that the cutoff scale is of order $\sim 10^{15}$ GeV and $\sin^2 \theta_w \sim 0.21$ which is off by ten percent from the true value. We also have a prediction of the Higgs mass in the interval 170 – 180 GeV. This can be taken as an indication that the noncommutative structure of space-time reveals itself at such high scale where the effective action has a geometrical interpretation. The slight disagreement with experiment indicates that the spectrum of the standard model could not be extrapolated to very high energies without adding new particles necessary to change the RG equations of the gauge couplings. One possibility could be supersymmetry, but there could be also less drastic solutions. This also could be taken as an indication that the concept of space-time as a manifold breaks down and the noncommutativity of the algebra must be extended to include the manifold part. Finally, we hope that our universal action formula should be applicable to many physical systems of which the most important could be superconformal field theory where the supersymmetry charge plays the role of the Dirac operator.

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